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UNIQUENESS PROPERTIES OF CR-FUNCTIONS(1)

BY

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ABSTRACT. Let M be a real infinitely differentiable closed hypersurface in X, a complex manifold of complex dimension $n > 2$. The uniqueness properties of solutions to the system $\bar{\partial}_M u = f$, where $\bar{\partial}_M$ is the induced Cauchy-Riemann operator on M, are of interest in the fields of several complex variables and partial differential equations. Since $\bar{\partial}_M$ is linear, the study of the solution to the equation $\bar{\partial}_M u = 0$ is sufficient for uniqueness. A C^∞ solution to this homogeneous equation is called a CR-function on M. The main result of this article is that a CR-function is uniquely determined, at least locally, by its values on a real k-dimensional C^∞ generic submanifold S^k of M with $k > n$. The facts that S^k is generic and $k > n$ together form the lower dimensional analogue of the concept of noncharacteristic.

1. **Introduction.** Let M be a C^∞ real hypersurface contained in X, a complex manifold of complex dimension $n \geqslant 2$. We are interested in the uniqueness properties of solutions to the equation $\bar{\partial}_M u = 0$, where $\bar{\partial}_M$ is the induced Cauchy-Riemann operator on M. Much study has been devoted to this problem in the case that the Levi form on M vanishes at every point (e.g. see [9]). Recently, results were obtained in [5] concerning uniqueness of analytic continuation for C^∞ CR-functions on a C^∞ real k-dimensional CR-submanifold M^k of X under the assumption that the Levi algebra on M^k has maximal dimension. In the case of a connected hypersurface $M^{2n-1} = M$, the C^∞ CR-functions on M have unique continuation if the Levi form is nonvanishing everywhere on M. However, it was felt that unique continuation should not depend on the Levi form, but instead on the nonexistence of characteristics for $\bar{\partial}_M$ (which are complex submanifolds of M of dimension $(n-1)$, called complex hypersurfaces). This was verified in [4], where it was shown that CR-distributions on M have local unique continuation across noncharacteristic hypersurfaces of M. In fact, it is proved that the boundary of the support of a CR-distribution on M is foliated by complex hypersurfaces. Of particular interest to us in [4] is the following result: If S is a C^1 real hypersurface in M and u is a continuous CR-distribution on M which vanishes on S, then u vanishes identically in an open

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neighborhood of each point $p \in S$ at which S is noncharacteristic.

Suppose S^k is a C^∞ real k-dimensional CR-submanifold of M and p is a point in S^k. Are there conditions on S^k at p which will ensure us that any C^∞ CR-function on M which vanishes on S^k near p will also vanish in an open neighborhood of p in M? If there does not exist such an open neighborhood of p, is there some CR-submanifold ω of M, which contains S^k near p and which is of maximal dimension in some sense, such that all C^∞ CR-functions vanishing on S^k near p must vanish on ω? With regard to the first question, the answer is affirmative if S^k is generic near p and $k \geqslant n$. It is interesting to compare this result for CR-functions to the known corresponding result for holomorphic functions. If a holomorphic function on a connected open subset U of $\mathbf{C}^n$ vanishes on an $S^k \subset U$ which is generic and with $k \geqslant n$, then this function is the zero function on U. Returning to our consideration of CR-functions on M, examples of CR-submanifolds S^k of M which are nongeneric at p or satisfy $k < n$ and for which we cannot obtain vanishing in an open set containing p are given. We will also answer the second question affirmatively in certain cases where S^k near p is contained in a lower dimensional complex subspace of X.

We have the following application of the above results. Let f be an $(n-1)$-tuple of C^∞ functions defined in an open neighborhood U of a point $p \in M$ and suppose that we have a C^∞ solution to the system $\bar{\partial}_M u = f$ in U. Any two such solutions which agree on a C^∞ totally real (i.e. having no complex tangent vectors) submanifold S^k of M with $k = n$ and $p \in S^k$ must agree on an open neighborhood of p in M.

The main tools used in our analysis of the above-mentioned problems are the solution of the additive Riemann-Hilbert problem of Andreotti and Hill [1] (or, more generally, the recent work of Polking and Wells on generalized boundary values [8]) and the general CR-extension theory found in [7]. The uniqueness results of [4] are referred to often. Familiarity with these references is helpful in understanding this work.

2. **Uniqueness properties.** We begin by giving some definitions and our notation to be used in this article.

Let X be a complex manifold of complex dimension $n \geqslant 2$, and let $M \subset X$ be a C^∞ real hypersurface in X. Suppose $T(M)$ is the real tangent bundle to M with fiber $T_p(M)$ for $p \in M$, and suppose $H(M)$ is the holomorphic tangent bundle to M with fiber $H_p(M)$. Denote by $\bar{\partial}_M$ the Cauchy-Riemann operator on M which is induced by the operator $\bar{\partial}$ on X. A complex-valued C^∞ function f on M for which $\bar{\partial}_M f = 0$ on M is a *CR-function* on M. If u is a distribution on M which satisfies $\bar{\partial}_M u = 0$ on M in the distribution sense, then u is a *CR-distribution* on M.

It is shown in [4] that a C^1 submanifold S of M of real dimension $(2n-2)$ is *characteristic at* $p \in S$ for $\bar{\partial}_M$ if and only if $T_p(S) = H_p(M)$, where $T_p(S)$ is the real tangent space to S at p. Also S is a *characteristic* surface (i.e. S is characteristic at each of its points) if and only if $T(S) = H(M)|_S$.

Let S^k be a C^∞ real k-dimensional submanifold of X, and define $T(S^k)$, $T_p(S^k)$, $H(S^k)$, and $H_p(S^k)$ for $p \in S^k$ as we did for M. Then S^k is *locally CR* at p if $\dim_{\mathbf{C}} H_X(S^k)$ is a constant for all $x \in S^k$ near p, and S^k is *locally generic* at p if $\dim_{\mathbf{C}} H_X(S^k) = \max(k-n, 0)$ for all $x \in S^k$ near p. If $k \leqslant n$ and S^k is locally generic at p, we say that S^k is *totally real* at p. If S^k has one of these respective properties globally, then we have a *CR-manifold, generic manifold*, or *totally real manifold*. A C^∞ function h on S^k satisfying $\bar{\partial}_{S^k} h = 0$ on S^k is a *CR-function* on S^k, with $\bar{\partial}_{S^k}$ being the induced Cauchy-Riemann operator on S^k. Similarly, we have a *CR-distribution* on S^k. The *Levi form* on a CR-manifold S^k is a quadratic form which essentially measures the lack of integrability of $H(S^k) \otimes \mathbf{C}$.

Let V be an open subset of our hypersurface M, and let p be a point in V. Suppose that the level surface $S = \{x \in V | r(x) = 0\}$ is noncharacteristic for $\bar{\partial}_M$ at p, where $r \in C^1(V)$ satisfies $r(p) = 0$ and $dr(p) \neq 0$. It is shown in [4] that there is a neighborhood ω of p in M such that if u is a continuous CR-distribution in V and $u \equiv 0$ on S, then $u \equiv 0$ in ω.

We are interested in proving results of this type with S being replaced by an appropriate lower dimensional C^∞ submanifold S^k of M and with u being replaced by a C^∞ CR-function on M. A real hypersurface S in M is noncharacteristic at a point $p \in S$ with respect to $\bar{\partial}_M$ if and only if it is generic at p. If S^k is a real k-dimensional submanifold of M and $k < n$, then the possibility of having a result analogous to the above mentioned theorem for S seems to be remote. The following examples show us that to hope for some type of general theorem we must have S^k generic at $p \in S^k$ and $k \geqslant n$.

EXAMPLE 1. Let U be an open set in X and suppose M is defined in U as the zero set of the C^∞ real-valued function ρ, which has nonvanishing gradient on U. Let S^k be a real k-dimensional CR-submanifold of $U \cap M$ which is defined in U as the common zero set of the C^∞ real-valued functions $\rho, \phi_1, \ldots, \phi_{2n-k-1}$ with linearly independent gradients in U. Suppose that $k \geqslant n$ and the complex fiber dimension of $H(S^k)$ is $(k-n)+d$ for some integer $d > 0$, thus making S^k nongeneric. Assume that $\phi_1 + i\phi_2, \phi_3 + i\phi_4, \ldots, \phi_{2d-1} + i\phi_{2d}$ are holomorphic functions in U. These functions are CR-functions on M which vanish on S^k, but which do not vanish on any open neighborhood in M of any point $p \in S^k$. Moreover, the common zero set in M of these holomorphic functions is a C^∞ real $(2(n-d)-1)$-dimensional submanifold of M which contains S^k. It is interesting to note that we will later prove that all CR-functions on M which vanish on S^k must vanish on a C^∞ real $(2(n-d)-1)$-dimensional CR-submanifold of M containing S^k.

EXAMPLE 2. We take U, M, and ρ as in Example 1. Let S^k be a C^∞ real k-dimensional generic submanifold of $U \cap M$ with $k < n$. Suppose that S^k is defined in U as the common zero set of the C^∞ real-valued functions $\rho, \phi_1, \ldots, \phi_{k-1}$ and the holomorphic functions (in U) $h_1, h_2, \ldots, h_{(n-k)}$. We assume, of course, that $\rho, \phi_1, \ldots, \phi_{k-1}$, Re h_1, Im $h_1, \ldots$, Re $h_{(n-k)}$, Im $h_{(n-k)}$ have linearly independent gradients in U. Then the functions $h_1, h_2, \ldots, h_{(n-k)}$ are CR-functions on M which vanish on S^k, but not on any open neighborhood in M of any point $p \in S^k$. It will be shown later that there exists a C^∞ real $(2k-1)$-dimensional CR-submanifold of M containing S^k such that all CR-functions on M that vanish on S^k must vanish on this manifold.

These examples illustrate that the following theorem is the best possible result of its type for C^∞ submanifolds S^k of M and CR-functions on M.

2.1. THEOREM. *Let $V \subset M$ be open and let $p \in V$. Suppose that S^k is a C^∞ real k-dimensional submanifold of V which is locally generic at $p \in S^k$ and with $k \geqslant n$. Then there is a neighborhood ω of p in M such that if f is a CR-function on V and $f \equiv 0$ on S^k, then $f \equiv 0$ in ω.*

PROOF. Without loss of generality we can assume that $V = U \cap M$, where U is an open Stein neighborhood of p in X. Also we can choose U such that S^k is generic in U. We assume that U is divided by M into disjoint open sets $\mathring{U}^+$ and $\mathring{U}^-$, and we let $U^+ = \mathring{U}^+ \cup V$ and $U^- = \mathring{U}^- \cup V$. By Theorem 1 of [1] there exist a C^∞ function f^+ defined in U^+ satisfying $\bar\partial f^+ = 0$ in U^+ and a C^∞ function f^- defined in U^- satisfying $\bar\partial f^- = 0$ in U^- such that $f = f^+ - f^-$ on M. Since $f \equiv 0$ on S^k and $f = f^+ - f^-$ on M we have $f^+ = f^-$ on S^k.

Suppose V is defined as the set $\{x \in U | \rho(x) = 0\}$ where $\rho \in C^\infty(U)$ and $d\rho$ does not vanish in U. Also assume that S^k is defined in U as the set $\{x \in U | \rho(x) = 0, \ \phi_1(x) = 0, \ldots, \phi_{2n-k-1}(x) = 0\}$ where $\rho, \phi_1, \ldots, \phi_{2n-k-1} \in C^\infty(U)$ and where $d\rho, d\phi_1, \ldots, d\phi_{2n-k-1}$ are linearly independent in U. Let N be the C^∞ real $(k+1)$-dimensional generic submanifold of U defined by $N = \{x \in U | \phi_1(x) = 0, \ldots, \phi_{2n-k-1}(x) = 0\}$, and notice that $M \cap N = S^k$.

Since S^k is generic at p there exists a complex vector $w \in H_p(N)$ such that $w \notin H_p(M)$. We define a new C^∞ real $(k+1)$-dimensional submanifold of U as the set $\{x \in U | \phi_1(x) = 0, \phi_2(x) = 0, \ldots, \phi_{2n-k-1}(x) = 0\}$, where $\tilde\phi_1 = \phi_1 + A\rho^2$ and where A is a positive constant still to be chosen. To conserve notation we call this new set N and remark that N satisfies $M \cap N = S^k$. Calculating, we find $\partial\tilde\phi_1 = \partial\phi_1 + 2A\rho\partial\rho$ and $\partial\bar\partial\tilde\phi_1 = \partial\bar\partial\phi_1 + 2A\bar\partial\rho \wedge \partial\rho +$

$2A\rho\partial\bar{\partial}\rho$. Now $\langle\partial\bar{\partial}\tilde{\phi}_1, w \wedge \bar{w}\rangle = \langle\partial\bar{\partial}\phi_1, w \wedge \bar{w}\rangle + 2A|\langle\partial\rho, w\rangle|^2$. Since $\langle\partial\rho(p), w\rangle \neq 0$, we have that $\langle\partial\bar{\partial}\tilde{\phi}_1(p), w \wedge \bar{w}\rangle$ is positive for A chosen large enough, implying that the Levi form on N at p is nonzero.

We define a function $\tilde{f}$ on N by

$$\tilde{f}(x) = \begin{cases} f^+(x), & x \in U^+ \cap N, \\ f^-(x), & x \in U^- \cap N. \end{cases}$$

This $\tilde{f}$ is a continuous function on N, and by Theorem 4.2 of [3], it is a continuous CR-distribution on N. We will leave the proof that $\tilde{f}$ is actually a CR-function (our CR-functions are C^∞) on N to a lemma following this proof.

By Theorem 5.1 of [7] the function $\tilde{f}$ extends to a CR-function $\tilde{\tilde{f}}$ on a real $(k+2)$-dimensional submanifold $\tilde{N}$ of U. Now $\tilde{N}$ consists of 1-dimensional analytic discs in some complex variable whose boundaries fill up a neighborhood of p in N (see [7]). Also $\tilde{N}$ is a generic manifold which is connected and simply connected. From [7] we have that $\tilde{N}$ has a C^m differentiable structure, where m can be taken as large as we wish depending on the size of $\tilde{N}$ and the Sobolev Lemma. This is enough differentiability for our purpose, but we understand that C. Denson Hill has recently shown that $\tilde{N}$ can be taken to be C^∞. Since the discs are constructed with respect to the vector $w \in H_p(N)$ satisfying $w \notin H_p(M)$, arbitrarily close to p we can find an open set Ω of discs, each of which intersects M transversally. Because $f^+ = \tilde{f}$ on that part of the boundary of each of these discs, which is in U^+ and because f^+ and $\tilde{f}$ are holomorphic in U^+ in the complex variable defining the discs, we have that $f^+ = \tilde{f}$ on the intersection of Ω with U^+. Similarly, we can show that $f^- = \tilde{f}$ on the intersection of Ω with U^-. Hence $f^+ = f^-$ on an open subset of $M \cap \tilde{N}$ arbitrarily close to p and $f \equiv 0$ on this open set. As the remainder of the proof will show, we can assume without loss of generality that $f \equiv 0$ on $M \cap \tilde{N}$.

Thus we have shown that f vanishes on a real $(k+1) =$ dimensional generic submanifold $M \cap \tilde{N}$ of M (if one takes local equations for M, S^k, N, and $\tilde{N}$, it is not difficult to see that N being generic implies that $M \cap \tilde{N}$ is also generic). Now we can choose a point $p' \in M \cap \tilde{N}$, which can be taken arbitrarily close to p, and repeat the argument. In $(2n-k-1)$ steps we find an open subset W of M such that $f \equiv 0$ on W. Since W can be chosen as close to p as we wish, we have that either $f \equiv 0$ in an open neighborhood ω of p in M or $p \in \partial(\operatorname{supp} f)$, the boundary of the support of f. We will show that it is impossible that $p \in \partial(\operatorname{supp} f)$, thus completing our proof.

By Theorem 4.2 of [4], the boundary of the support of f is foliated by complex hypersurfaces, and these complex hypersurfaces are nowhere dense and nonintersecting. If $p \in \partial(\operatorname{supp} f)$, then p is contained in some complex hypersurface $S \subset \partial(\operatorname{supp} f)$ and we assume $S \subset U$. Since S^k is generic in U and since $k \geqslant n$, any complex hypersurface that intersects S^k in U must do so transversally in M.

Therefore V is an open neighborhood of p in M which contains a set A consisting of open sets having as their boundaries complex hypersurfaces $\subset \partial(\operatorname{supp} f)$ which disconnect V and the appropriate parts of ∂V. If Y is one of these open sets which intersects S^k, then the intersection consists of generic points of S^k which are not in $\partial(\operatorname{supp} f)$, and by a previous step in our proof, $f \equiv 0$ on Y. Since S and S^k intersect transversally, if one works in a sufficiently small open neighborhood ω of p in V, then the intersections with ω of all open sets in A are like Y (where these intersections are nonempty) and $f \equiv 0$ on ω. Q.E.D.

The notation in the following lemma is that contained in the proof of Theorem 2.1.

2.2. LEMMA. *The function $\tilde{f}$ defined as f^+ in $U^+ \cap N$ and f^- in $U^- \cap N$ is a CR-function on N.*

PROOF. We need only show that $\tilde{f}$ is C^∞ on N at S^k. Let p be an arbitrary point in S^k and let U be an open neighborhood of p in X. Suppose the coordinates for X near p are $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$, which vanish at p, and suppose the local coordinates for S^k at p are $x_1, y_1, \ldots, x_{k-n}, y_{k-n}, x_{k-n+1}, x_{k-n+2}, \ldots, x_n$, also vanishing at p. For the local coordinates on N at p which vanish at p we take

$$x_1, y_1, \ldots, x_{k-n}, y_{k-n}, x_{k-n+1}, x_{k-n+2}, \ldots, x_n, y_n.$$

It is very important that x_n and y_n are paired in this way, and this is essentally the "noncharacteristic aspect" that makes the lemma valid. We assume that S^k near p is given by

$$\begin{aligned} &z_j = x_j + iy_j, \qquad j = 1, \ldots, k-n, \\ &z_j = x_j + ih_{j-k+n}(x_1, y_1, \ldots, x_{k-n}, y_{k-n}, \\ &\qquad\qquad x_{k-n+1}, x_{k-n+2}, \ldots, x_n), \\ &\qquad\qquad\qquad j = k-n+1, \ldots, n, \end{aligned} \tag{2.1}$$

with each h_{j-k+n} vanishing to order 2 at p.

Using the fact that $\bar{\partial} f^{\pm} = 0$ on $U^{\pm}$ and computing by the chain rule at S^k from $U^{\pm}$ we find that

$$\begin{bmatrix}
1 & 0 & \cdots & 0 & \dfrac{\partial z_{k-n+1}}{\partial x_1} & \cdots & \dfrac{\partial z_n}{\partial x_1} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & & \vdots \\
0 & 0 & \cdots & 1 & \dfrac{\partial z_{k-n+1}}{\partial x_{k-n}} & \cdots & \dfrac{\partial z_n}{\partial x_{k-n}} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \dfrac{\partial z_{k-n+1}}{\partial x_{k-n+1}} & \cdots & \dfrac{\partial z_n}{\partial x_{k-n+1}} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \dfrac{\partial z_{k-n+1}}{\partial x_n} & \cdots & \dfrac{\partial z_n}{\partial x_n} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \dfrac{\partial z_{k-n+1}}{\partial y_1} & \cdots & \dfrac{\partial z_n}{\partial y_1} & 1 & 0 & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \dfrac{\partial z_{k-n+1}}{\partial y_{k-n}} & \cdots & \dfrac{\partial z_n}{\partial y_{k-n}} & 0 & 0 & \cdots & 1
\end{bmatrix}$$

$$\begin{bmatrix}
\dfrac{\partial f^{\pm}}{\partial x_1} \\ \vdots \\ \dfrac{\partial f^{\pm}}{\partial x_{k-n}} \\ \dfrac{\partial f^{\pm}}{\partial x_{k-n+1}} \\ \vdots \\ \dfrac{\partial f^{\pm}}{\partial x_n} \\ \dfrac{\partial f^{\pm}}{\partial y_1} \\ \vdots \\ \dfrac{\partial f^{\pm}}{\partial y_{k-n}}
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{\partial f^{\pm}}{\partial x_1} \\ \vdots \\ \dfrac{\partial f^{\pm}}{\partial x_{k-n}} \\ \dfrac{\partial f^{\pm}}{\partial x_{k-n+1}} \\ \vdots \\ \dfrac{\partial f^{\pm}}{\partial x_n} \\ \dfrac{\partial f^{\pm}}{\partial y_1} \\ \vdots \\ \dfrac{\partial f^{\pm}}{\partial y_{k-n}}
\end{bmatrix}_t$$

where $[\cdot]_t$ denotes tangential derivatives. Since f^+ and f^- agree on S^k, all tangential derivatives of first order of f^+ and f^- agree on S^k. Since the coefficient matrix in the above system is nonsingular, we have that all corresponding first order derivatives of f^+ and f^- are equal on S^k (i.e. $\partial f^+/\partial x_1 = \partial f^-/\partial x_1$, $\partial f^+/\partial y_1 = \partial f^-/\partial y_1$, etc.).

An induction argument shows that the corresponding derivatives of f^+ and f^- with respect to $x_1, y_1, \ldots, x_{k-n}, y_{k-n}, x_{k-n+1}, x_{k-n+2}, \ldots, x_n$ of all orders agree on S^k.

Let α be a multi-index of any order with

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_{2(k-n)-1}, \alpha_{2(k-n)}, \alpha_{2(k-n)+1}, \ldots, \alpha_k, \alpha_{k+1})$$

corresponding to the variables $x_1, y_1, x_2, y_2, \ldots, x_{k-n}, y_{k-n}, x_{k-n+1}, \ldots, x_n, y_n$. Since $\bar{\partial} f^\pm = 0$ in $U^\pm$, we have $D^\alpha f^\pm = i^{\alpha_{k+1}} D^{\alpha'} f^\pm$ on S^k, where D^α is the appropriate partial derivative according to the multi-index notation, $i = \sqrt{-1}$, and

$$\alpha' = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_{2(k-n)-1}, \alpha_{2(k-n)}, \alpha_{2(k-n)+1}, \ldots, \alpha_k + \alpha_{k+1}, 0).$$

However, $D^{\alpha'} f^+ = D^{\alpha'} f^-$ on S^k, implying that $D^\alpha f^+ = D^\alpha f^-$ on S^k. Thus the corresponding derivatives and tangential derivatives of f^+ and f^- with respect to the coordinates of N are equal at every point of S^k. Q.E.D.

We have the following corollary to Theorem 2.1. For a precise description of the foliation mentioned, see [4].

2.3. COROLLARY. *Suppose that M is connected and contains no complex hypersurface, the closure of which is foliated by complex hypersurfaces. If S^k is a C^∞ real k-dimensional submanifold of M with $k \geqslant n$ which is generic at some point $p \in S^k$, then any CR-function f on M which vanishes on S^k near p must vanish identically on M.*

PROOF. By Theorem 2.1 the function f vanishes in an open neighborhood of p in M. Then Theorem 4.3 of [4] implies that M has global unique continuation for its CR-functions, and hence $f \equiv 0$ on M. Q.E.D.

John Polking has pointed out that a connected compact $M \subset X$ must satisfy the hypothesis of Corollary 2.3. Thus we have the following global result for such a hypersurface in X.

2.4. COROLLARY. *Let M be a connected and compact real hypersurface in X. Then any CR-function f on M which vanishes on a C^∞ real k-dimensional generic submanifold S^k of M with $k \geqslant n$ must vanish identically on M.*

Suppose now that S^k is a C^∞ real k-dimensional CR-submanifold of an open set in X and suppose that $k \geqslant n$. Let S^k be contained in M and assume that S^k is nongeneric. It is well known that

$$\max(k-n, 0) \leqslant \dim_{\mathbf{C}} H_p(S^k) \leqslant [k/2]$$

for every point $p \in S^k$. We set

$$d = \dim_{\mathbf{C}} H_p(S^k) - \max(k - n, 0)$$

and call d the *exceptionality* of S^k. Essentially d measures the extent to which S^k is nongeneric.

Let p be some arbitrary point in S^k and let U be an open coordinate neighborhood of p in X. It is shown in [6] that there exists a C^∞ real k-dimensional generic submanifold $T^k \subset \mathbf{C}^{n-d}$ such that S^k is the graph of d functions $g_1, \ldots, g_d$ defined on T^k with $p' \in T^k$ mapping to $p \in S^k$. If T^k has complex structure then it is true that $g_1, \ldots, g_d$ are *CR*-functions on T^k. This T^k is the *associated generic manifold* to S^k near p. Suppose in the local equation for S^k near p we have the defining functions $z_{n-d+1} - g_1 = 0, \ldots, z_n - g_d = 0$, with $g_1, \ldots, g_d$ defined in U and vanishing to order 2 at p. Assume that near p, M is defined as the zero set of the C^∞ function ρ with $d\rho = (\partial\rho/\partial y_{n-d})dy_{n-d} \neq 0$ when evaluated at p.

2.5. THEOREM. *Suppose that $k \geqslant n - d$ and $g_1, \ldots, g_d$ are holomorphic functions in U. If f is a CR-function on M which vanishes on S^k near p, then there exists a C^∞ real $(2(n-d)-1)$-dimensional CR-submanifold $\omega \subset M$ which contains S^k near p and such that f vanishes on ω. There exists no manifold of real dimension greater than $2(n-d)-1$ which contains S^k near p, so that all such f vanish on this manifold.*

PROOF. In the local coordinates for X at p, replace $z_{n-d+1}, \ldots, z_n$ by $z_{n-d+1} - g_1, \ldots, z_n - g_d$, respectively. In these new coordinates, S^k near p is simply a generic manifold in $\mathbf{C}^{n-d}$ with $k \geqslant n - d$. An application of Theorem 2.1 proves the first conclusion in the theorem. The functions $z_{n-d}, \ldots, z_n$ (in the new coordinates) are *CR*-functions on M which vanish on S^k near p and prove the last statement of the theorem. Q.E.D.

Let S^k be a C^∞ real k-dimensional generic submanifold of an open set in X and let $k < n$. Suppose $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$ are the local coordinates in some U for X near p and $x_1, \ldots, x_k$ are the local coordinates for S^k near p. Assume that M is defined near p as the zero set of the function $y_k - h(x_1, y_1, \ldots, x_k, x_{k+1}, y_{k+1}, \ldots, x_n, y_n)$. The set S^k can be represented as the graph over a C^∞ k-dimensional generic (totally real in this case) submanifold T^k of $\mathbf{C}^k$ with the C^∞ functions $g_1, \ldots, g_{n-k}$ defining the graph.

2.6. THEOREM. *Suppose that $g_1, \ldots, g_{n-k}$ are holomorphic functions in U. If f is a CR-function on M which vanishes on S^k near p, then there exists a C^∞ real $(2k-1)$-dimensional CR-submanifold $\omega \subset M$ which contains S^k near p and such that f vanishes on ω. There exists no manifold of real dimension greater than $(2k-1)$, which contains S^k near p, such that all such f vanish on this manifold.*

PROOF. The proof of this result is analogous to that of Theorem 2.5. Q.E.D.

REMARK. A result similar to the last two theorems concerning nongeneric CR-submanifolds of dimension k in X with $k < n$ could also be proved.

REMARK. C. Denson Hill has shown the author that our principal result, Theorem 2.1, can be proved by applying Theorem 1 of [1], Lemma 2.2, and the "edge of the wedge" results of Eric Bedford [2]. This is a consequence of the fact that the important case in Theorem 2.1 is $k = n$.

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